Transverse nature of optical wave

Again, let's use the Euler formula:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

$$E(\bar{z}, t) = A\,\mathrm{Re}\left[e^{i(\omega t - \bar{k}\bar{z})}\right]$$

Now, let's prove that e/m wave is transverse (opposite to Huygens idea)

$$\frac{\partial \bar{E}}{\partial t} = i\omega \bar{E} \qquad \frac{\partial E}{\partial z} = -i\bar{k}\bar{E}$$

$$\begin{cases} \operatorname{div} \bar{E} = 0 \\ \operatorname{div} \bar{B} = 0 \end{cases} \Rightarrow \begin{cases} \bar{k}\bar{E} = 0 \\ \bar{k}\bar{B} = 0 \end{cases}$$

$$\begin{cases} c^2 \operatorname{rot} \bar{B} - \frac{\partial \bar{E}}{\partial t} = 0 \\ \operatorname{rot} \bar{E} + \frac{\partial \bar{B}}{\partial t} = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} c^2[\bar{k}\bar{B}] + \omega \bar{E} = 0 \\ [\bar{k}\bar{E}] - \omega \bar{B} = 0 \end{cases} \Rightarrow \begin{cases} c^2[\bar{k}\bar{B}] = -\omega \bar{E} \\ [\bar{k}\bar{E}] = \omega \bar{B} \end{cases}$$

$\bar{k}\bar{E} = 0 \rightarrow \bar{k}$ and $\bar{E}$ are perpendicular

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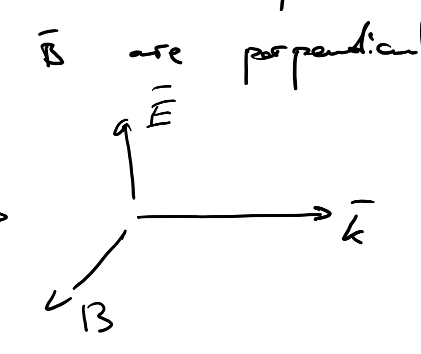
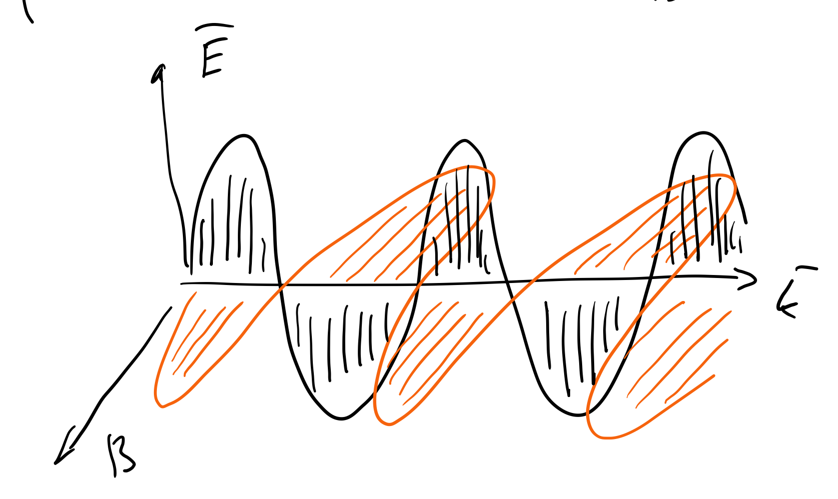
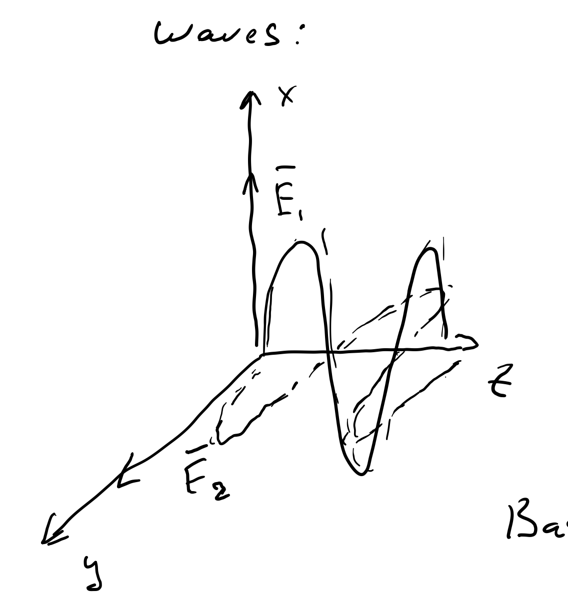
$$\begin{cases} [\bar{k}\bar{E}] = \omega \bar{B} \\ c^2[\bar{k}\bar{B}] = -\omega \bar{E} \end{cases} \rightarrow$$





$\bar{E}\bar{k}$ plane is called polarization plane.
When $\bar{E}$ is in one polarization plane, then the light is *linearly polarized*.

Let's take two perpendicular polarized waves:



$$\bar{E}_1(E_{01}\cos(\omega t - kz), 0, 0)$$

$$\bar{E}_2(0, E_{02}\cos(\omega t - kz + \varphi), 0)$$

φ – phase difference between them

Based on superposition principle:

$$\boxed{\bar{E} = \bar{E}_1 + \bar{E}_2}$$

$$\bar{E}(\underbrace{E_{01}\cos(\omega t - kz)}_{E_x}, \underbrace{E_{02}\cos(\omega t - kz + \varphi)}_{E_y}, 0)$$

First, we need to get rid of t:

1) $E_x = E_{01}\cos(\omega t - kz)$

$$\frac{E_x}{E_{01}} = \cos(\omega t - kz) \Rightarrow \sqrt{1 - \left(\frac{E_x}{E_{01}}\right)^2} = \sin(\omega t - kz)$$

2) $E_y = E_{02}\cos(\omega t - kz + \varphi)$

$$\frac{E_y}{E_{02}} = \cos(\omega t - kz + \varphi) = \cos(\omega t - kz)\cos\varphi - \sin(\omega t - kz)\sin\varphi$$

$$\Downarrow$$

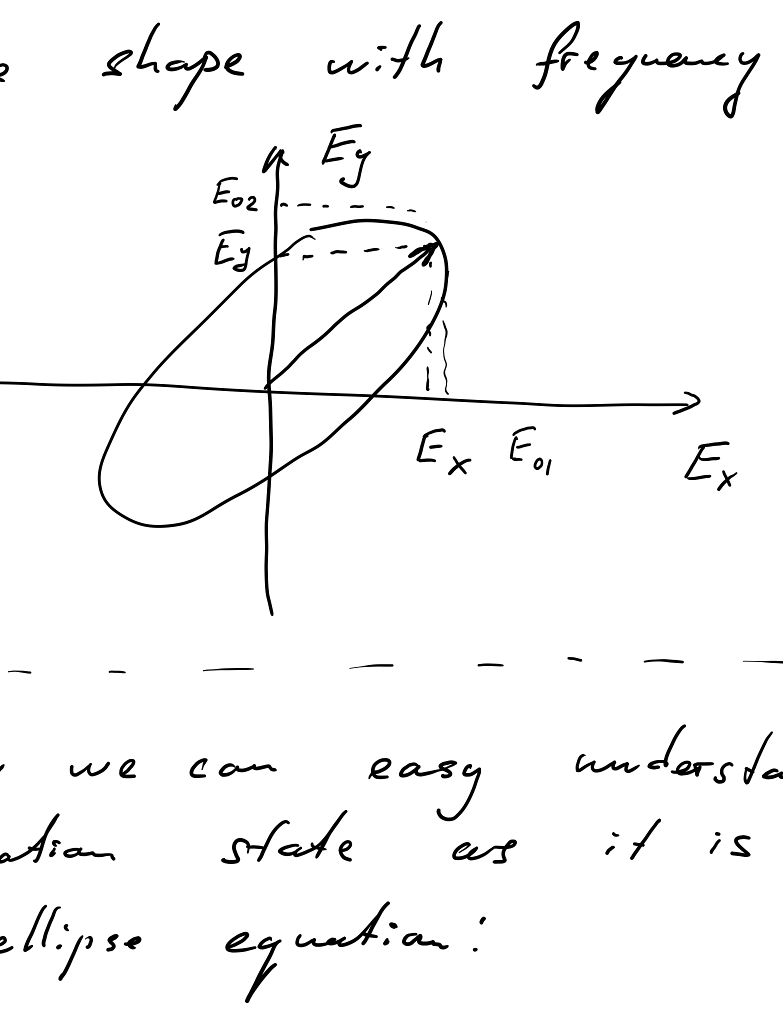
$$\frac{E_y}{E_{02}} = \frac{E_x}{E_{01}}\cos\varphi - \sqrt{1 - \left(\frac{E_x}{E_{01}}\right)^2}\sin\varphi$$

$$\left(\frac{E_y}{E_{02}}\right)^2 - 2\left(\frac{E_x}{E_{01}}\right)\left(\frac{E_y}{E_{02}}\right)\cos\varphi + \left(\frac{E_x}{E_{01}}\right)^2 = \sin^2\varphi$$

This is ellipse equation!

What does it mean?

The end of $\bar{E}$ vector is following ellipse shape with frequency ω

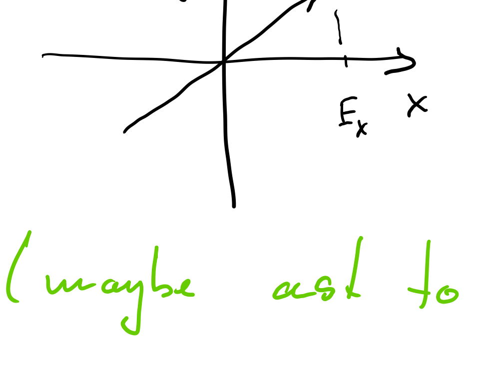


Now we can easy understand each polarization state as it is special case of ellipse equation:

① $\varphi = 0$

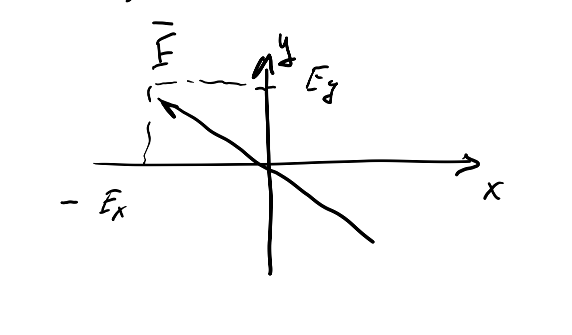
$$\left(\frac{E_y}{E_{02}} - \frac{E_x}{E_{01}}\right)^2 = 0 \Rightarrow E_y = \frac{E_{02}}{E_{01}}E_x,$$

i.e. linear



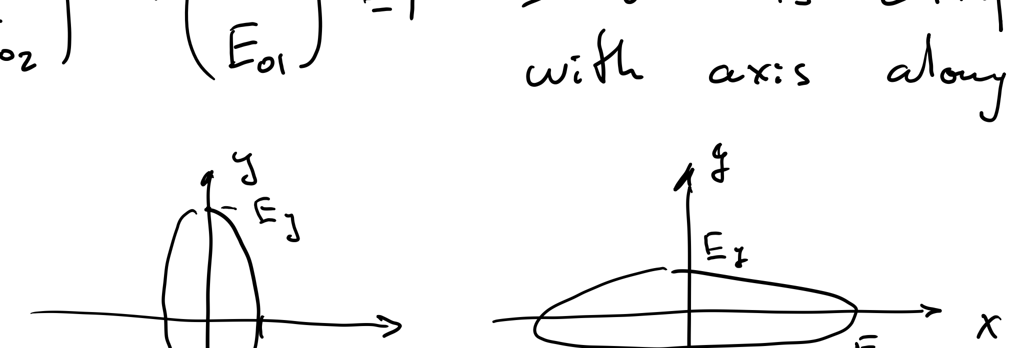
② $\varphi = \pi$ (maybe ask to do it)

$$\left(\frac{E_y}{E_{02}} + \frac{E_x}{E_{01}}\right)^2 = 0 \Rightarrow E_y = -\frac{E_{02}}{E_{01}}E_x$$

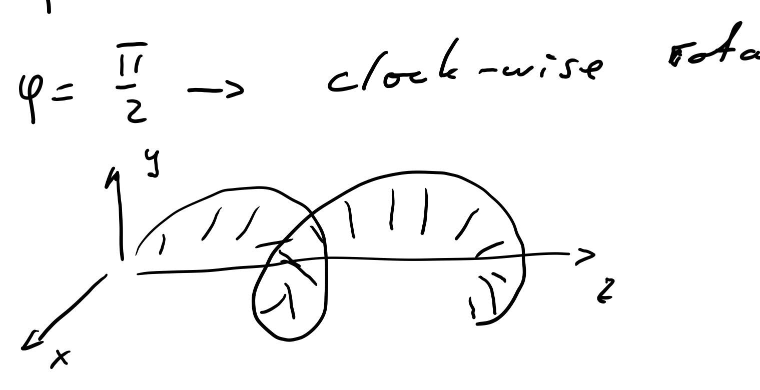


③ $\varphi = \frac{\pi}{2}$

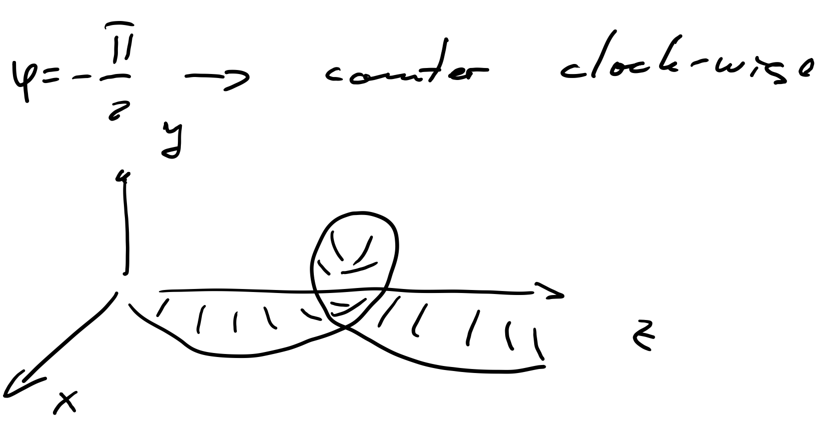
$$\left(\frac{E_y}{E_{02}}\right)^2 + \left(\frac{E_x}{E_{01}}\right)^2 = 1$$ → this is ellipse equation with axis along x and y



If $\varphi = \frac{\pi}{2}$ → clock-wise rotation

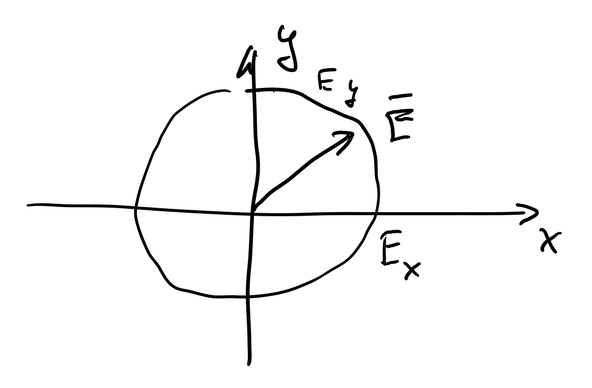


If $\varphi = -\frac{\pi}{2}$ → counter clock-wise rotation

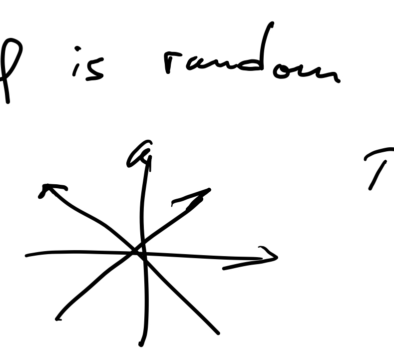


④ $\varphi = \frac{\pi}{2}$, $E_{01} = E_{02}$

$$E_y^2 + E_x^2 = E_{01}^2$$ → this is circle equation



⑤ φ is random



This is natural or unpolarized light.